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A multiplicity result for a class of elliptic boundary value problems

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Synopsis

We consider a mildly nonlinear elliptic boundary value problem depending on a parameter. Given appropriate hypotheses concerning the asymptotic behaviour of the nonlinearity, we derive lower bounds on the number of solutions. The results complement an earlier theorem due to Kazdan and Warner [**6**].

I. Statement of the Result

We consider the semilinear elliptic boundary value problem (BVP)

$$(P_t)\quad \begin{aligned} Au &= f(x, u; t) \quad && \text{in } \Omega, \\ Bu &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where t is a real parameter. Here Ω is a bounded domain in R^N $(N \geqq 1)$ with smooth boundary $\partial\Omega$, and

$$Au := -\sum_{j,k=1}^{N} a_{jk} D_j D_k u + \sum_{j=1}^{N} a_j D_j u + a_0 u$$

a second order linear elliptic differential operator with smooth real coefficients and a uniformly positive definite coefficient matrix (a_{jk}). Further B denotes either the Dirichlet or the Neumann boundary operator. Let λ_1 be the principal eigenvalue of the linear BVP

$$\begin{aligned} Au &= \lambda u \quad && \text{in } \Omega, \\ Bu &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The following hypotheses are imposed on the nonlinearity f:

(f1) the function f: $\bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is smooth;

(f2) for every $m \in \mathbb{R}$, there exists a function $h \in C(\bar{\Omega})$ such that $D_3 f(x, \xi; t) \geqq$

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$h(x)>0$ for all $x\in\Omega$, $\xi\geqq m$, and $t\in R$;

(f3) for every $x\in\bar\Omega$, $t\in\mathbb{R}$,

$$\text{(f3}') \ \limsup_{\xi\to-\infty}\frac{f(x,\xi;t)}{\xi}<\lambda_1$$

and

$$\text{(f3}'') \ \liminf_{\xi\to+\infty}\frac{f(x,\xi;t)}{\xi}>\lambda_1,$$

uniformly for $x\in\bar\Omega$ and t in bounded intervals;

$$\text{(f4)} \ \limsup_{\xi\to+\infty}\frac{f(x,\xi;t)}{\xi}<+\infty,$$

uniformly for $x\in\bar\Omega$ and t in bounded intervals.

We are now in position to state our main result.

THEOREM. *Under the above hypotheses, there exists a number $t_0\in\mathbb{R}$ such that (P_t) has no (classical) solution if $t>t_0$, at least one solution if $t=t_0$, and at least two distinct solutions if $t<t_0$.*

Some remarks concerning the comparison of this result with related former research are in order.

1. It has been shown by Kazdan and Warner [**6**, Corollary 3.11] that hypotheses (f1)–(f3) (without the uniformity assumption with respect to t) imply the existence of a number $t_0\in\mathbb{R}$ such that (P_t) has no solution if $t>t_0$ and at least one solution if $t<t_0$. No multiplicity result is obtained there, and no assertion is made for $t=t_0$.

2. Suppose f is of the special form

$$f(x,\xi;t)=f_0(x)+tr(x)+g(x,\xi), \tag{1}$$

with $r(x)>0$ for $x\in\Omega$. Then f satisfies hypotheses (f2)–(f4) provided

$$\limsup_{\xi\to-\infty}\frac{g(x,\xi)}{\xi}<\lambda_1$$

and

$$\lambda_1<\liminf_{\xi\to+\infty}\frac{g(x,\xi)}{\xi}\leqq\limsup_{\xi\to+\infty}\frac{g(x,\xi)}{\xi}<+\infty,$$

uniformly for $x\in\bar\Omega$. In this case, besides the assertion of the Theorem, its proof further yields the closedness in $C(\bar\Omega)$ of the set of functions p for which the nonlinear *BVP*

$$\begin{cases} Au=p(x)+g(x,u) & \text{in } \Omega,\\ Bu=0 & \text{on } \partial\Omega \end{cases}$$

has a solution. We generalize the result of Ambrosetti and Prodi [**3**] and Berger and Podolak [**4**]. Recall that in [**4**], for formally selfadjoint A, with B the Dirichlet boundary operator, $g(x,\xi)=g(\xi)$ and $r=\varphi$ (the positive eigenfunction to the eigenvalue λ_1), it is shown that there exists $t_0 \in \mathbb{R}$ such that (P_t) has no solution for $t > t_0$, precisely one solution for $t = t_0$, and exactly two solutions for $t < t_0$, provided

$$0 < g'(-\infty) < \lambda_1 < g'(+\infty) < \lambda_2$$

and g is strictly convex.

3. Our theorem also sharpens a recent result of Hess and Ruf [**5**]. In that paper, for formally selfadjoint A and f of the form (1) with $r = \varphi$, it is assumed that

$$\lim_{\xi \to -\infty} \frac{g(x, \xi)}{\xi} = -\infty, \qquad \lim_{\xi \to +\infty} \frac{g(x, \xi)}{\xi} > \lambda_1$$

(uniformly in $x \in \bar{\Omega}$), and the existence of two constants $T_1 \leqq T_2$ is asserted such that problem (P_t) admits no solution for $t > T_2$ and at least two solutions for $t < T_1$ (perform the change of variable $u \to -u$ in order to bring the problem considered in [**5**] to the present setting).

Since we rely on [**6**, Corollary 3.11], we suppose (as in [**6**]) that B is either the Dirichlet or the Neumann boundary operator. However, it is not difficult to verify that everything remains true if B is the boundary operator associated with the third *BVP*, i.e.

$$Bu = \frac{\partial u}{\partial \beta} + b_0 u,$$

where $b_0 \geqq 0$ and β is an outward pointing (nowhere tangent) smooth vectorfield on $\partial\Omega$.

II. Proof of the Theorem

(a) It follows from [**6**, Corollary 3.11] that there exists a real number t_0 such that (P_t) has no solution for $t > t_0$ and at least one solution for $t < t_0$. Let $t^* < t_0$ be fixed, and choose $\tau \in (t^*, t_0)$. Then there exists a smoooth function $\bar{u}$ such that

$$\begin{cases} A\bar{u} = f(x, \bar{u}; \tau) & \text{in } \Omega, \\ B\bar{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since, by hypothesis (f2), f is strictly increasing in the variable t, it follows that $\bar{u}$ is a strict supersolution for (P_{t^*}). By means of hypothesis (f3) and the arguments in [**6**, Lemma 2.7], we can find a strict subsolution $\underline{u}$ of (P_{t^*}) with $B\underline{u} = 0$, such that $\underline{u} < \bar{u}$.

(b) Let

$$\omega_0 := \max\{|D_2 f(x, \xi; t^*)| : x \in \bar{\Omega}, \min \underline{u} \leqq \xi \leqq \max \bar{u}\},$$

and set

$$\omega := \max\{\omega_0 + 1, \|a_0\|_{C(\bar{\Omega})}\}.$$

Moreover, let

$$F(u, t)(x) := f(x, u(x); t) + \omega u(x),$$

for all $(x, t) \in \bar{\Omega} \times \mathbb{R}$ and all functions $u: \bar{\Omega} \to \mathbb{R}$, and denote by Kv the unique solution of the linear BVP

$$\begin{cases} (A+\omega)u = v & \text{in } \Omega, \\ \qquad Bu = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, let

$$E := C_B^1(\bar{\Omega}) := \{u \in C^1(\bar{\Omega}) : Bu = 0 \text{ on } \partial\Omega\},$$

equipped with the norm $\|\cdot\|$ of $C^1(\bar{\Omega})$, and endow all function spaces with their natural order. Then it is well-known [e.g. **2**, Section 4] that $K: C(\bar{\Omega}) \to E$ is compact and strongly positive, and that problem (P_t) is equivalent to the fixed point equation

$$u = KF(u, t)$$

in E. The mapping

$$KF: E \times \mathbb{R} \to E$$

is continuous, and maps bounded sets into relatively compact sets. We note that

$$\underline{u} < KF(\underline{u}, t^*) \quad \text{and} \quad KF(\bar{u}, t^*) < \bar{u}, \tag{2}$$

and that $K \mp (\cdot, t^*)$ is strongly increasing on the order interval

$$X := [\underline{u}, \bar{u}] := \{u \in E : \underline{u} \leqq u \leqq \bar{u}\}.$$

Since X is bounded in $C(\bar{\Omega})$, it follows that $\overline{KF(X, t^*)}$ is compact in E [cf. **2,** Section 9].

(c) Set $G := KF(\cdot, t^*)$. We want to show that G has at least two distinct fixed points. Since G is increasing, (2) implies that $G(X) \subset X$. Then, by Schauder's fixed point theorem, G has a fixed point u_0 in X. Since G is strongly increasing, (2) further implies that X has nonempty interior $\mathring{X}$, and that $u_0 \in \mathring{X}$. Of course we can assume that u_0 is the only fixed point of G in X (otherwise we are done). Then there exists $\varepsilon > 0$ such that $u_0 + \varepsilon\mathbb{B} \subset X$ (where $\mathbb{B}$ is the open unit ball in E), and such that the Leray–Schauder degree

$$\deg(id - G, u_0 + \varepsilon\mathbb{B}, 0)$$

is defined. By making use of the uniqueness of the Leray-Schauder degree and the permanence and excision properties of the fixed point index [cf. **2**, Theorem (11.1) and the first formula in its proof], we find that

$$\begin{aligned} \deg(id - G, u_0 + \varepsilon\mathbb{B}, 0) &= i(G, u_0 + \varepsilon\mathbb{B}, E) \\ &= i(G, u_0 + \varepsilon\mathbb{B}, X) = i(G, X, X) = 1. \end{aligned} \tag{3}$$

Here the last equality is a trivial consequence of the convexity of X and the homotopy invariance property (cf. the proof of the Schauder fixed point theorem in [**2**, p. 660]).

(d) Suppose

$$\begin{cases} \text{there exists } \rho > 0 \text{ such that } u_0 + \varepsilon\mathbb{B} \subset \rho\mathbb{B} \text{ and} \\ KF(u, t) \neq u \text{ for all } t \in I := [t^*, t_0 + 1] \\ \text{and all } u \in E \text{ with } \|u\| = \rho. \end{cases} \tag{4}$$

Then, by the homotopy invariance of the Leray-Schauder degree,

$$\deg(id - G, \rho\mathbb{B}, 0) = \deg(id - KF(\cdot, t_0 + 1), \rho\mathbb{B}, 0) = 0,$$

since, according to the definition of t_0, $KF(\cdot, t_0 + 1)$ has no fixed point at all in E. Thus, by (3),

$$\deg(id - G, \rho\mathbb{B}\backslash(u_0 + \varepsilon\bar{\mathbb{B}}), 0) = \deg(id - G, \rho\mathbb{B}, 0) - \deg(id - G, u_0 + \varepsilon\mathbb{B}, 0) = -1,$$

which implies that there is a fixed point of G in $\rho\mathbb{B}\backslash(u_0 + \varepsilon\bar{\mathbb{B}})$. Hence the existence of at least two distinct solutions of problem (P_{t^*}) is proved provided we verify the *a priori* estimate (4).

(e) Observe that K can be looked upon as a strongly positive compact endomorphism of E. Thus the spectral radius $r(K)$ is positive and the only eigenvalue of K having a positive eigenvector [e.g. **2**, Theorem (3.2)]. It follows that $r(K) = (\lambda_1 + \omega)^{-1}$.

Hypothesis (f3) implies that there exist numbers $\mu < \lambda_1 + \omega$ and $k \geqq 0$ such that

$$F(u, t) \geqq \mu u - k \tag{5}$$

for all $u: \bar{\Omega} \to \mathbb{R}$ and all $t \in I$. Let w be the unique solution of the linear equation

$$w - \mu K w = -kK\mathbb{1}.$$

Then (5) implies that

$$(u_t - w) - \mu K(u_t - w) \geqq 0$$

for every fixed point u_t of $KF(\cdot, t)$, $t \in I$. Consequently, since $1/\mu > r(K)$, [**2**, Theorem 3.2 (iv)] implies

$$u_t \geqq w \quad \text{for every} \quad t \in I. \tag{6}$$

Suppose now (4) is not true. Then we find sequences (t_j) in I and $(u_j := u_{t_j})$ in E with $\|u_j\| \to \infty$, such that

$$u_j = KF(u_j, t_j) \tag{7}$$

for all $j \in \mathbb{N}$. Let $v_j := u_j/\|u_j\|$ and observe that hypothesis (f4) and (6) imply that $\{F(u_j, t_j)/\|u_j\|: j \in \mathbb{N}\}$ is bounded in $C(\bar{\Omega})$. Dividing (7) by $\|u_j\|$ and using the compactness of K as a map from $C(\bar{\Omega})$ to E, it follows that the sequence (v_j) is relatively compact in E. Hence, by passing to an appropriate subsequence, we may assume that

$$v_j \to v \quad \text{in } E$$

where, due to (6),

$$v \geqq 0. \tag{8}$$

Hypothesis (f3) implies also the existence of numbers $\alpha > 0$, $\beta \geqq 0$ such that

$$F(u, t) \geqq (\lambda_1 + \omega + \alpha)u - \beta$$

for all $u: \bar{\Omega} \to \mathbb{R}$ and all $t \in I$. Thus, by (7),

$$v_j \geqq (\lambda_1 + \omega + \alpha)Kv_j - \|u_j\|^{-1}\beta K\mathbb{1},$$

and in the limit we get

$$v-(\lambda_1+\omega+\alpha)Kv\geqq 0.$$

Since $(\lambda_1+\omega+\alpha)^{-1}<r(K)$, [**2**, Theorem 3.2 (iv)] and (8) imply $v=0$. But this contradicts the obvious fact that $\|v\|=1$.

(f) In order to prove that problem (P_t) admits a solution also for $t=t_0$, we take a sequence $(t_j)\uparrow t_0$. It then follows from part (e) of the present proof that corresponding solutions u_j:

$$u_j=KF(u_j, t_j)\quad (j\in\mathbb{N})$$

remain bounded in E. Next we observe that the sequence (u_j) is relatively compact in E; for a suitable subsequence we have $u_j\to u$ in E and $u=KF(u, t_0)$. Thus u is a solution of (P_{t_0}). $\square$

III. An extension

An inspection of the above proof and the fact that there is no uniformity assumption with respect to t in the hypothesis of [**6**, Corollary 3.11] shows that the following more precise result is true.

Proposition. *Suppose there exists a number T such that hypotheses* (f3″) *and* (f4) *hold only for $t\geqq T$, and that the generalized limits in* (f3) *and* (f4) *are uniform only for t in bounded intervals of $[T, +\infty)$. Then there exists $t_0\in R$ such that problem (P_t) has no solution for $t>t_0$ and at least one solution for $t<t_0$. If $t_0>T$, then (P_t) has at least two distinct solutions for $T\leqq t<t_0$ and at least one solution for $t=t_0$.*

The following example shows that this result is in some sense optimal, i.e. that in general we cannot expect two solutions for $t<T$.

Example. Suppose $f:\mathbb{R}\to\mathbb{R}^+$ is a smooth, increasing, strictly convex function satisfying

$$f'(-\infty)=0\quad\text{and}\quad \lambda_1<f'(+\infty)<+\infty$$

(where $\lambda_1>0$ is the first eigenvalue of $-\Delta$, subject to Dirichlet boundary conditions). Moreover, assume that

$$f(\xi)<\xi f'(\xi)$$

for sufficiently large $\xi>0$. Consider the *BVP*

$$(A_t)\begin{cases}-\Delta u=tf(u) & \text{in }\Omega,\\ \quad u=0 & \text{on }\partial\Omega.\end{cases}$$

Then it follows from [**6**, Corollary 3.11] that there exists a $t_0\in\mathbb{R}$ such that (A_t) has no solution for $t>t_0$ and at least one solution for $t<t_0$. Moreover, [**2**, Theorems (20.12) and (26.3)] imply that $t_0>0$, and that there exists a number $t_\infty\in(0, t_0)$ such that (A_t) has at least two solutions for $t_\infty<t<t_0$ and exactly one solution for $t=t_0$ and $t\in[0, t_\infty]$. The monotonicity of f implies further that (A_t) has exactly one solution for $t<0$. Finally, t_∞ is the principal eigenvalue of the linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda f'(+\infty)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e. $t_\infty = \lambda_1 / f'(+\infty)$. Observe that, for every $T > t_\infty$, problem (A_t) satisfies the assumptions of the above Proposition, but not for $T \leqq t_\infty$.

Lastly we remark that the Proposition generalizes [**1**, Theorem (7.6)], [cf. also **2**, Section 21].

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Note added in proof, 31 July 1979. After this paper had been submitted for publication the authors became aware of an article by E. N. Dancer, On the ranges of certain weakly nonlinear elliptic partial differential equations, *J. Math. Pure Appl.* **57** (1978), 351–366. In that paper a similar result is proved by closely related considerations.

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